Metode de simulare a sistemelor analogice cu sisteme discrete

Problema 1. Fie sistemul analogic liniar şi invariant în timp descris de funcția de transfer:

$$H_C(s)=\frac{s+2}{(s+1)^2(s+3)}.$$

a) Să se determine răspunsul său la impuls, $h_C(t)$;

b) Să se determine răspunsul la impuls al sistemului numeric echivalent pe baza metodei invarianței răspunsului la impuls $h_d[n]$;

c) Să se determine $H_d(z)$, transformata z a lui $h_d[n]$;

d) Să se obțină ecuația cu diferențe finite pentru sistemul discret echivalent sistemului analogic considerat.

Rezolvare.

Punctul a)

$$H_c(s)=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s+3}$$

$$B=(s+1)^2H_c(s)\Big|_{s=-1}=\frac{s+2}{s+3}\Big|_{s=-1}=\frac{1}{2}\Rightarrow B=\frac{1}{2}$$

$$C=(s+3)H_c(s)\Big|_{s=-3}=\frac{s+2}{(s+1)^2}\Big|_{s=-3}=-\frac{1}{4}$$

$$A=\frac{d}{ds}\left[(s+1)^2H_c(s)\right]\Big|_{s=-1}=\frac{d}{ds}\left[\frac{s+2}{s+3}\right]\Big|_{s=-1}=\frac{s+3-s-2}{(s+3)^2}\Big|_{s=-1}=\frac{1}{4}$$

$$H_c(s)=\underbrace{\frac{1}{4}\cdot\frac{1}{s+1}}_{\substack{\updownarrow\\ \frac{1}{4}e^{-t}\sigma(t)}}+\frac{1}{2}\cdot\frac{1}{(s+1)^2}-\underbrace{\frac{1}{4}\cdot\frac{1}{s+3}}_{\substack{\updownarrow\\ -\frac{1}{4}e^{-3t}\sigma(t)}}$$

$$-\frac{1}{(s+1)^2}=\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

$$x(t)\leftrightarrow X(s)\Rightarrow\frac{dX(s)}{ds}=\int_{-\infty}^{\infty}x(t)(-t)e^{-st}dt=-\int_{-\infty}^{\infty}tx(t)e^{-st}=-\mathcal{L}\{tx(t)\}(s)$$

$$-\frac{1}{(s+1)^2}=-L\{te^{-t}\sigma(t)\}(s)\Rightarrow-\frac{1}{(s+1)^2}\leftrightarrow-te^{-t}\sigma(t)$$

$$h_c(t)=\frac{1}{4}e^{-t}\sigma(t)+\frac{1}{2}te^{-t}\sigma(t)-\frac{1}{4}e^{-3t}\sigma(t)$$

Punctul b) $h_d\left[n\right]=h_c\left(nT\right)=\frac{1}{4}e^{-nT}\sigma\left[n\right]+\frac{T}{2}ne^{-nT}\sigma\left[n\right]-\frac{1}{4}e^{-3nT}\sigma\left[n\right]$

Punctul c)

$$\frac{1}{4}e^{-nT}\sigma\left[n\right]=\frac{1}{4}\left(e^{-T}\right)^n\sigma\left[n\right]\leftrightarrow\frac{1/4}{1-e^{-T}z^{-1}}$$

$$-\frac{1}{4}e^{-3nT}\sigma\left[n\right]=-\frac{1}{4}\left(e^{-3T}\right)^n\sigma\left[n\right]\leftrightarrow-\frac{1/4}{1-e^{-3T}z^{-1}}$$

$$\frac{T}{2}ne^{-nT}\sigma\left[n\right]=\frac{T}{2}n\left(e^{-T}\right)^n\sigma\left[n\right]\leftrightarrow\frac{T}{2}\frac{e^{-T}z^{-1}}{(1-e^{-T}z^{-1})^2}$$

$$H_d\left(z\right)=\frac{1/4}{1-e^{-T}z^{-1}}+\frac{Te^{-T}}{2}\cdot\frac{z^{-1}}{(1-e^{-T}z^{-1})^2}-\frac{1/4}{1-e^{-3T}z^{-1}}$$

Punctul d)

$$H_d\left(z\right)=\frac{\frac{1}{4}\left(1-e^{-T}z^{-1}\right)\left(1-e^{-3T}z^{-1}\right)+\frac{Te^{-T}}{2}z^{-1}\left(1-e^{-3T}z^{-1}\right)-\frac{1}{4}\left(1-e^{-T}z^{-1}\right)^2}{\left(1-e^{-T}z^{-1}\right)^2\left(1-e^{-3T}z^{-1}\right)}=$$

$$=\frac{\frac{1}{4}-\frac{1}{4}\left(e^{-T}+e^{-3T}\right)z^{-1}+\frac{e^{-4T}}{4}z^{-2}+\frac{Te^{-T}}{2}z^{-1}-\frac{Te^{-4T}}{2}z^{-2}-\frac{1}{4}+\frac{1}{2}e^{-T}z^{-1}-\frac{1}{4}e^{-2T}z^{-2}}{}=$$

$$=\frac{z^{-2}\left(\frac{e^{-4T}}{4}-\frac{Te^{-4T}}{2}-\frac{1}{4}e^{-2T}\right)+z^{-1}\left(\frac{Te^{-T}}{2}-\frac{1}{4}e^{-T}-\frac{1}{4}e^{-3T}+\frac{1}{2}e^{-T}\right)}{1-2e^{-T}z^{-1}+e^{-2T}z^{-2}-e^{-3T}z^{-1}+2e^{-4T}z^{-2}-e^{-5T}z^{-3}}$$

$$H_d\left(z\right)=\frac{\frac{e^{-4T}-2Te^{-4T}-e^{-2T}}{4}z^{-2}+\frac{2e^{-T}+2Te^{-T}-e^{-T}-e^{-3T}}{4}z^{-1}}{-e^{-5T}z^{-3}+\left(e^{-2T}+2e^{-4T}\right)z^{-2}-\left(2e^{-T}+e^{-3T}\right)z^{-1}+1}$$

$$H_d\left(z\right)=\frac{\left(e^{-4T}-2Te^{-4T}-e^{-2T}\right)z^{-2}+\left(2Te^{-T}+e^{-T}-e^{-3T}\right)z^{-1}}{-4e^{-5T}z^{-3}+4e^{-2T}\left(1+2e^{-2T}\right)z^{-2}-4e^{-T}\left(2+e^{-2T}\right)z^{-1}+4}$$

$$H_d\left(z\right)=\frac{b_1z^{-2}+b_1z^{-1}}{a_3z^{-3}+a_2z^{-2}+a_1z^{-1}+a_0}$$

$$b_2=e^{-2T}\left(e^{-2T}-2Te^{-2T}\right);b_1=e^{-T}\left(2T+1-e^{-2T}\right);b_0=0$$

$$a_3=-4e^{-5T};a_2=4e^{-2T}\left(1+2e^{-2T}\right);a_1=-4e^{-T}\left(2+e^{-2T}\right);a_0=4$$

Ecuația cu diferențe finite este :

$$-4e^{-5T}y\left[n-3\right]+4e^{-2T}\left(1+2e^{-2T}\right)y\left[n-2\right]-4e^{-T}\left(2+e^{-2T}\right)y\left[n-1\right]+4=$$
$$e^{-T}\left(2T+1-e^{-2T}\right)x\left[n-1\right]+e^{-2T}\left(e^{-2T}-2Te^{-2T}-1\right)x\left[n-2\right]$$

Problema 2. Fie $H_c(s)$ funcția de sistem a unui sistem analogic liniar şi invariant în timp, $H_c(s)=\dfrac{1}{(s+1)(s+2)^2}$.

a) Să se determine răspunsul la impuls al sistemului analogic, $h_c(t)$;

b) Se notează cu $s_c(t)$ răspunsul indicial al sistemului analogic. Se notează cu $h_d[n]$ răspunsul la impuls şi cu $s_d[n]$ răspunsul indicial al unui sistem discret liniar şi invariant îin timp. În ipoteza că sistemul discret s-a obținut prin echivalarea sistemului analogic considerat folosind metoda de echivalare bazată pe invarianța răspunsului la impuls, adică $h_d[n]=h_c(nT)$, este adevarată relația :

$$s_d[n]=\sum_{k=-\infty}^{n} h_c(kT) \quad ?$$

c) Dacă sistemul discret echivalent s-a obținut prin metoda invarianței răspunsului indicial, adică $s_d[n]=s_c(nT)$, este adevarată relația : $h_d[n]=h_c(nT)$?

d) In ipoteza de la punctul c), a invarianței răspunsului indicial, să se determine $H_d(z)$ pentru sistemul discret echivalent.

Rezolvare

Punctul a)

$$H_c(s)=\frac{1}{(s+1)(s+2)^2}=\frac{A}{s+1}+\frac{B}{(s+2)^2}+\frac{C}{s+2}$$

$$A=(s+1)H_c(s)\Big|_{s=-1}=\frac{1}{(s+2)^2}\Big|_{s=-1}=1$$

$$B=(s+2)^2H_c(s)\Big|_{s=-2}=\frac{1}{s+1}\Big|_{s=-2}=-1$$

$$C=\frac{d}{ds}\Big[(s+2)^2H_c(s)\Big]\Big|_{s=-2}=\frac{d}{ds}\left(\frac{1}{s+1}\right)\Big|_{s=-2}=-\frac{1}{(s+1)^2}\Big|_{s=-2}=-1$$

$$h_c(t)=e^{-t}\sigma(t)-te^{-2t}\sigma(t)-e^{-2t}\sigma(t)$$

Punctul b)

Metoda 1

Fie $S_c(s)$ transformata Laplace a raspunsului indicial al sistemului analogic.

$$S_c(s)=\frac{H_c(s)}{s}=\frac{1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+2)^2}+\frac{D}{s+2}$$

Se calculează răspunsul indicial al sistemului analogic :

$$A=sS_c(s)\Big|_{s=0}=\frac{1}{(s+1)(s+2)}\Big|_{s=0}=\frac{1}{4}$$

$$B=(s+1)S_c(s)\Big|_{s=-1}=\frac{1}{s(s+2)^2}\Big|_{s=-1}=-1$$

$$C=(s+2)^2 S_c(s)\Big|_{s=-2}=\frac{1}{s(s+1)}\Big|_{s=-2}=\frac{1}{2}$$

$$D=\frac{d}{ds}\Big[(s+2)^2 S_c(s)\Big]\Big|_{s=-2}=\frac{d}{ds}\left[\frac{1}{s^2+s}\right]\Big|_{s=-2}=-\frac{2s+1}{\left(s^2+s\right)}\Big|_{s=-2}=-\frac{-3}{4}=3/4$$

$$s_c(t)=\frac{1}{4}\sigma(t)-e^{-t}\sigma(t)+\frac{1}{2}e^{-2t}\sigma(t)+\frac{3}{4}e^{-2t}\sigma(t)$$

$$h_d[n]=h_c(nt)=e^{=nT}\sigma[n]-nTe^{-2nT}\sigma[n]-e^{-2nT}\sigma[n]$$

$$h_d[n]=(e-T)\sigma[n]-Tn\left(e^{-T}\right)^n\sigma[n]-\left(e^{-2T}\right)^n\sigma[n]$$

$$\sum_{k=-\infty}^{n}h_c(kT)=\sum_{k=-\infty}^{n}e^{-kT}\sigma(kT)-kTe^{-2kT}\sigma(kt)-e^{-2kT}\sigma(kT)=$$

$$=\sum_{k=0}^{n}e^{-kT}-T\sum_{k=0}^{n}ke^{-2kT}-\sum_{k=0}^{n}e^{-2kT}=\sum_{k=0}^{n}\left(e^{-T}\right)^k-T\sum_{k=0}^{n}k\left(e^{-2T}\right)^k-\sum_{k=0}^{n}\left(e^{-2T}\right)k$$

$$\sum_{k=-\infty}^{n}h_c(kT)=\sum_{k=-\infty}^{n}e^{-kT}\sigma(kT)-kTe^{-2kT}\sigma(kt)-e^{-2kT}\sigma(kT)=$$

$$=\sum_{k=0}^{n}e^{-kT}-T\sum_{k=0}^{n}ke^{-2kT}-\sum_{k=0}^{n}e^{-2kT}=\underbrace{\sum_{k=0}^{n}(e-T)^k}_{\substack{\Updownarrow\\ \frac{1-\left(e^{-T}\right)^{n+1}}{1-e^{-T}}}}-T\sum_{k=0}^{n}k\left(e^{-2T}\right)^k-\underbrace{\sum_{k=0}^{n}\left(e^{-2T}\right)^k}_{\substack{\Updownarrow\\ \frac{1-\left(e^{-2T}\right)^{n-1}}{-2T}}}$$

$$\sum_{k=0}^{n}\left(e^{-2T}\right)^k=\frac{1-\left(e^{-2T}\right)^{n+1}}{1-e^{-2T}}$$

Fie $2T=x\Rightarrow\sum_{k=0}^{n}e^{-kx}=\frac{1-\left(e^{-x}\right)^{n+1}}{1-e^{-x}}$

Derivând ultima relație în raport cu x se obține :

$$-\sum_{k=0}^{n}ke^{-kx}=\frac{(n+1)\left(e^{-x}\right)^n\left(1-e^{-x}\right)-e^{-x}\left[1-\left(e^{-x}\right)^{n+1}\right]}{\left(1-e^{-x}\right)^2}$$

Adică: $\sum_{k=0}^{n}ke^{-kx}=\frac{(n+1)e^{-nx}-(n+1)e^{-(n+1)x}-e^{-x}+\left(e^{-x}\right)^{n+2}}{\left(1-e^{-x}\right)^2}$

Pentru $x=2T$ se obține :

$$\sum_{k=0}^{n} k\left(e^{-2T}\right)^{k} = \frac{(n+1)e^{-2nT} - (n+1)e^{-2(n+1)T} - e^{-2T} + e^{-2(n+2)T}}{\left(1-e^{-2T}\right)^{2}}$$

Deci :

$$\sum_{k=-\infty}^{n} h_c(kT) =$$

$$\frac{1-e^{-(n+1)T}}{1-e^{-T}} - T\frac{(n+1)e^{-2nT} - (n+1)e^{-2(n+1)T} - e^{-2T} + e^{-2(n+2)T}}{\left(1-e^{-2T}\right)^{2}} - \frac{1-e^{-2(n+1)T}}{1-e^{-2T}}$$

$$= \frac{\left(1-e^{-2T}\right)^{2}\left(1-e^{-(n+1)T}\right) - (n+1)Te^{-2nT} + (n+1)Te^{-2(n+1)T}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}} +$$

$$+ \frac{Te^{-2T} - Te^{-2(n+2)T} - \left(1-e^{-2(n+1)T}\right)\left(1-e^{-T}\right)\left(1-e^{-2T}\right)}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}}$$

$$= \frac{\left(1-2e^{-2T}+e^{-4T}\right)\left(1-e^{-(n+1)T}\right) - (n+1)Te^{-2nT} + (n+1)Te^{-2(n+1)T}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}} +$$

$$+ \frac{Te^{-2T} - Te^{-2(n+2)T} - \left(1-e^{-T}-e^{-2T}+e^{-3T}\right)\left(1-e^{-2(n+1)T}\right)}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}}$$

$$= \frac{1-e^{-(n+1)T} - 2e^{-2T} + 2e^{-(n+3)T} + e^{-4T} - e^{-(n+5)T} - (n+1)Te^{-2nT}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}} +$$

$$+ \frac{(n+1)Te^{-2(n+1)T} + Te^{-2T} - Te^{-2(n+2)T} - 1 + e^{-T} + e^{-2T} - e^{-3T}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}} +$$

$$+ \frac{e^{-2(n+1)T} - e^{-2\left(n+\frac{3}{2}\right)T} - e^{-2(n+2)T} + e^{-2\left(n+1+\frac{3}{2}\right)T}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^{2}}$$

Adică :

$$\begin{aligned}\sum_{k=-\infty}^{n} h_c(kT) &= \frac{e^{(-2n-5)T} - e^{(-2n-4)T} - e^{(-2n-3)T} + e^{(-2n-2)T}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^2} + \\ &+ \frac{-Te^{(-2n-4)T} + (n+1)Te^{(-2n-2)T} - (n+1)Te^{-2nT}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^2} + \\ &+ \frac{-e^{(-n-5)T} + 2e^{(-n-3)T} - e^{(-n-1)T} - (2+T+1)e^{-2T} + e^{-4T} - e^{-3T}}{\left(1-e^{-T}\right)\left(1-e^{-2T}\right)^2}\end{aligned} \tag{1}$$

$$H_d(z) = \frac{1}{1-e^{-T}\cdot z^{-1}} - T\frac{e^{-T}\cdot z^{-1}}{\left(1-e^{-T}\cdot z^{-1}\right)^2} - \frac{1}{1-e^{-2T}\cdot z^{-1}}$$

$$= \frac{\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right) - Te^{-T}\cdot z^{-1}\left(1-e^{-2T}\cdot z^{-1}\right)}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)} +$$

$$+ \frac{-\left(1-e^{-T}\cdot z^{-1}\right)^2}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)}$$

$$= \frac{1-e^{-T}\cdot z^{-1} - e^{-2T}\cdot z^{-1} + e^{-3T}\cdot z^{-2} - Te^{-T}\cdot z^{-1}}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)} +$$

$$+ \frac{+Te^{-3T}\cdot z^{-2} - \left(1+e^{-2T}\cdot z^{-2} - 2e^{-T}\cdot z^{-1}\right)}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)}$$

$$= \frac{z^{-2}\left(e^{-3T} + Te^{-3T} - e^{-2T}\right) - z^{-1}\left(e^{-T} + e^{-2T} + Te^{-T} + 2e^{-T}\right)}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)}$$

$$H_d(z) = \frac{e^{-2T}\left(e^{-T} + Te^{-T} - 1\right)z^{-2} - e^{-T}\left(3+T+e^{-T}\right)z^{-1}}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)}$$

$$S_d(z) = \frac{H_d(z)}{1-z^{-1}} = \frac{e^{-2T}\left(e^{-T} + Te^{-T} - 1\right)z^{-2} - e^{-T}\left(3+T+e^{-T}\right)z^{-1}}{\left(1-e^{-T}\cdot z^{-1}\right)^2\left(1-e^{-2T}\cdot z^{-1}\right)\left(1-z^{-1}\right)}$$

$$= \frac{A}{1-z^{-1}} + \frac{B}{1-e^{-2T}\cdot z^{-1}} + \frac{C}{\left(1-e^{-T}\cdot z^{-1}\right)^2} \;\Rightarrow\; s_d[n]$$

Se constată, folosind relația (1), că într-adevar $s_d\left[n\right]=\sum\limits_{k=-\infty}^{n} h_c\left(kT\right)$.

Metoda 2

$$s_d\left[n\right]=\sigma\left[n\right]*h_d\left[n\right]=\sum_{k=-\infty}^{\infty} h_d\left[k\right]\sigma\left[n-k\right]=$$

$$=\sum_{k=-\infty}^{\infty} h_d\left[k\right]=\sum_{k=-\infty}^{\infty} h_c\left[kT\right]$$

Deci : $s_d\left[n\right]=\sum\limits_{k=-\infty}^{\infty} h_c\left[kT\right]$.

In consecință relația propusă este adevarată în cazul general.

Punctul c)

$$s_d\left[n\right]=s_c\left[n\right]\ \Rightarrow$$

$$s_d\left[n\right]=\frac{1}{4}\sigma\left[n\right]-e^{-nT}\sigma\left[n\right]+\frac{1}{2}nT\cdot e^{-2nT}\sigma\left[n\right]+\frac{3}{4}e^{-2nT}\sigma\left[n\right]$$

$$S_d\left(z\right)=\frac{1}{4}\cdot\frac{1}{1-z^{-1}}-\frac{1}{1-e^{-T}\cdot z^{-1}}+\frac{T}{2}\cdot\frac{e^{-2T}\cdot z^{-1}}{\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{3}{4}\cdot\frac{1}{1-e^{-2T}\cdot z^{-1}}$$

$$S_d\left(z\right)=\frac{\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2-4\left(1-z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}{4\left(1-z^{-1}\right)\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{2Te^{-2T}\cdot z^{-1}\left(1-z^{-1}\right)\left(1-e^{-T}\cdot z^{-1}\right)}{4\left(1-z^{-1}\right)\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{3\left(1-z^{-1}\right)\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)}{4\left(1-z^{-1}\right)\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$H_d\left(z\right)=\left(1-z^{-1}\right)S_d\left(z\right)=$$

$$=\frac{\left(1-e^{-T}\cdot z^{-1}\right)\left(1-2e^{-2T}\cdot z^{-1}+e^{-4T}\cdot z^{-2}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{-4\left(1-z^{-1}\right)\left(1-2e^{-2T}\cdot z^{-1}+e^{-4T}\cdot z^{-2}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}+$$

$$+\frac{2Te^{-2T}\left(z^{-1}-z^{-2}\right)\left(1-e^{-T}\cdot z^{-1}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{3\left(1-z^{-1}-e^{-T}\cdot z^{-1}+e^{-T}\cdot z^{-2}\right)\left(1-e^{-2T}\cdot z^{-1}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$H_d\left(z\right)=\frac{1\cdot 2e^{-2T}\cdot z^{-1}+e^{-4T}\cdot z^{-2}\cdot e^{-T}\cdot z^{-1}+2e^{-3T}z^{-2}-e^{-5T}z^{-3}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{-4\left(1+e^{-4T}z^{-2}-2e^{-2T}z^{-1}-z^{-1}e^{-4T}z^{-3}-2e^{-2T}z^{-2}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{2Te^{-2T}\left(z^{-1}-e^{-T}z^{-2}-z^{-2}+e^{-T}z^{-3}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{3\left(1-z^{-1}e^{-T}z^{-1}+e^{-T}z^{-2}-e^{-2T}z^{-1}-e^{-2T}z^{-2}+e^{-3T}z^{-2}-e^{-3T}z^{-3}\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$=\frac{e^{-3T}\left(-e^{-2T}+4e^{-T}+2T-3\right)z^{-3}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{z^{-2}\left[-3e^{-4T}+\left(5+2T\right)e^{-3T}+\left(5-2T\right)e^{-2T}+3e^{-T}\right]}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{\left(5+2T\right)e^{-3T}+z^{-1}\left(-4e^{-T}+\left(5+2T\right)e^{-2T}+1\right)}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

Deci:

$$H_d\left(z\right)=\frac{e^{-3T}\left(e^{-2T}+4e^{-T}+2T-3\right)z^{-3}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{\left[-3e^{-4T}+\left(5+2T\right)e^{-3T}+3e^{-T}\right]z^{-2}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{\left(-4e^{-T}+\left(5+2T\right)e^{-2T}+1\right)z^{-1}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

Adică:

$$H_d\left(z\right)=\frac{e^{-3T}\left(e^{-2T}+4e^{-T}+2T-3\right)z^{-3}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{e^{-T}\left[-3e^{-3T}+\left(5+2T\right)e^{-2T}+\left(5-2T\right)e^{-T}+3\right]z^{-2}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$+\frac{\overbrace{\left[-4e^{-T}+\left(5+2T\right)e^{-2T}+1\right]z^{-1}}^{\alpha}}{4\left(1-e^{-T}\cdot z^{-1}\right)\left(1-e^{-2T}\cdot z^{-1}\right)^2}$$

$$H_d(z)=\frac{A}{1-e^{-T}\cdot z^{-1}}+\frac{B}{\left(1-e^{-2T}\cdot z^{-1}\right)^2}+\frac{C}{1-e^{-2T}\cdot z^{-1}}$$

$$A=\left(1-e^{-T}\cdot z^{-1}\right)H_d(z)\Big|_{z^{-1}=e^{T}}$$

$$=\left.\frac{e^{-3T}\left(-e^{-2T}+4e^{-T}+2T-3\right)z^{-3}}{4\left(1-e^{-2T}\cdot z^{-1}\right)^2}\right|_{z^{-1}=e^{T}}$$

$$+\left.\frac{e^{-T}\left[-3e^{-3T}+(5+2T)e^{-2T}+(5-2T)e^{-T}+3\right]z^{-2}+2z^{-1}}{4\left(1-e^{-2T}\cdot z^{-1}\right)^2}\right|_{z^{-1}=e^{T}}$$

$$A=\frac{-e^{-8T}-4e^{-7T}+(2T-3)e^{-6T}-3e^{-2T}+4e^{T}}{4\left(1-e^{-T}\right)^2}$$

$$h_d[n]=A\left(e^{-T}\right)^n\sigma[n]+B(n+1)\left(e^{-2T}\right)^n\sigma[n]+C\left(e^{-T}\right)^n\sigma[n]$$

$$h_c(nT)=e^{-nT}\sigma[n]-nTe^{-2nT}\sigma[n]-e^{-2nT}\sigma[n]$$

Deoarece: $A\neq 1 \;\Rightarrow\; h_d[n]\neq h_c(nT)$ deci relația propusă nu este adevarată.

Problema 3. Se notează $h_c(t)$ răspunsul la impuls al unui sistem liniar, invariant în timp continuu şi cu $h_d[n]$ cel al unui sistem liniar, invariant în timp discret ce echivalează sistemul analogic prin metoda invarianței răspunsului la impuls.

a) Dacă $h_c(t)=e^{-t}\sigma(t)$ să se determine functia de transfer a filtrului analogic şi să se schițeze dependenta de frecvență a modulului său ;
b) Să se determine funcția de transfer a filtrului numeric ;
c) Să se schițeze caracteristica de modul a răspunsului în frecvență al sistemului discret.

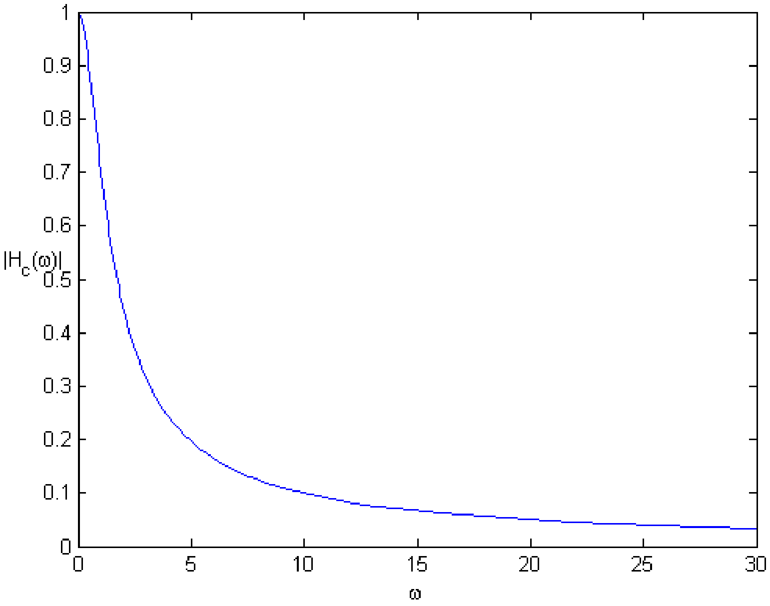
Rezolvare.

Punctul a)

$$H_c(s)=\mathcal{L}\left\{e^{-t}\sigma(t)\right\}=\frac{1}{s+1}$$

$$\Rightarrow H_c(\omega)=\frac{1}{1+j\omega}$$

$$\Rightarrow\left|H_c(\omega)\right|=\frac{1}{\sqrt{1+\omega^2}}$$



Punctul b)

$$k_d[n] = k_c(nT) = e^{-nT}\sigma[n] \;\Rightarrow\; H_d(z) = \frac{1}{1-e^{-T}z^{-1}}$$

Punctul c)

$$H_d(\Omega) = \frac{1}{1-e^{-T}\cdot e^{-j\Omega}} \quad \Rightarrow$$

$$|H_d(\Omega)| = \frac{1}{\left|1-e^{-T}\cdot\cos\Omega + je^{-T}\cdot\sin\Omega\right|}$$

$$= \frac{1}{\sqrt{\left(1-e^{-T}\cdot\cos\Omega\right)^2 + e^{-2T}\cdot\sin^2\Omega}}$$

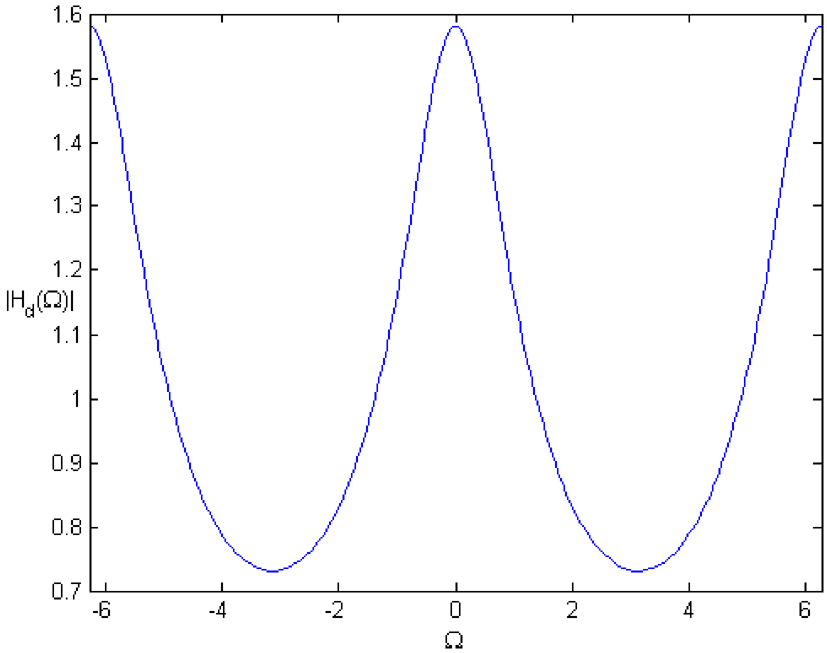
$$= \frac{1}{\sqrt{1+e^{-2T}-2e^{-T}\cos\Omega}}$$

$$|H_d(0)| = \frac{1}{1-e^{-T}} \cong \frac{1}{1-(1-T)} = \frac{1}{T}$$

$$\left|H_d\left(\frac{\pi}{2}\right)\right| = \frac{1}{\sqrt{1+e^{-2T}}}; \quad |H_d(\pi)| = \frac{1}{1+e^{-T}}$$

Pentru T=1

$$|H_d(0)| = 1; \; \left|H_d\left(\frac{\pi}{2}\right)\right| = \frac{1}{\sqrt{1+e^{-2}}} \cong 0{,}938; \; |H_d(\pi)| = \frac{1}{1+e^{-1}} = 0{,}73$$



Problema 4 Se consideră sistemul anlogic liniar şi invariant în timp descris de ecuaţia:

$$\frac{dy}{dt}+2y(t)=x(t)$$

a) Să se determine răspunsul la impuls al unui sistem discret ce simulează sistemul dat prin metoda invarianţei răspunsului la impuls, precum şi funcţia sa de sistem ;
b) Să se determine aceleaşi funcţii ca la punctul a) pentru un sistem discret echivalent, obţinut prin metoda aproximării numerice a ecuaţiei diferenţiale ;
c) Să se repete punctul a) pentru un sistem discret obţinut prin metoda transformării biliniare.
d) Schiţaţi formele de implementare pentru sistemele de la punctele a), b) şi c) ;
e) Pentru $x(t)=\sin t$ şi o perioada de esantionare $T=0{,}1$ să se determine amplitudinile semnalelor de ieşire ale sistemului analogic dat şi ale celor trei sisteme numerice.

Rezolvare

Punctul a)

$$\frac{dy}{dt}+2y(t)=x(t)\ \Rightarrow\ sY(s)+2Y(s)=X(s)\ \Rightarrow\ H_a(s)=\frac{Y(s)}{X(s)}=\frac{1}{s+2}$$

$$h_a(t)=e^{-2t}\sigma(t)\ \Rightarrow\ h_d[n]=e^{-2nT}\sigma[n]\ \Rightarrow\ H_d(z)=\frac{1}{1-e^{-2t}z^{-1}}$$

Punctul b)

$$H_d(z)=H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}=\frac{1}{2+\dfrac{1-z^{-1}}{T}}=\frac{1}{1+2T-z^{-1}}=\frac{\dfrac{T}{1+2T}}{1-\dfrac{1}{1+2T}\cdot z^{-1}}$$

$$h_d[n]=\frac{T}{1+2T}\left(\frac{1}{1+2T}\right)^n\sigma[n]$$

Punctul c)

$$H_d(z)=H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}=\frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+2}=\frac{1+z^{-1}}{2\left(1-z^{-1}+T\left(1+z^{-1}\right)\right)}$$

$$=\frac{1+z^{-1}}{2(1+T)-2(1-T)z^{-1}}=\frac{1}{2}\cdot\frac{1+z^{-1}}{1+T-(1-T)z^{-1}}$$

$$=\frac{1}{2}\cdot\frac{(1-T)\left[\left(z^{-1}-\frac{1+T}{1-T}\right)+1+\frac{1+T}{1-T}\right]}{(1-T)\left[1+T-(1-T)z^{-1}\right]}$$

$$=\frac{1}{2(1-T)}\cdot\frac{\left(z^{-1}-\frac{1+T}{1-T}\right)(1-T)}{(1-T)\left(\frac{1+T}{1-T}-z^{-1}\right)}+\frac{1}{2}\cdot\frac{1+\frac{1+T}{1-T}}{1+T-(1-T)z^{-1}}$$

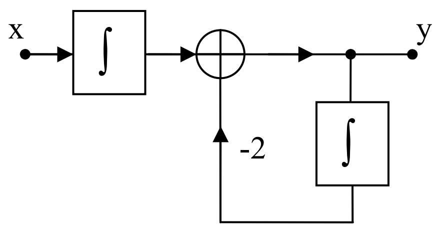
$$=\frac{-1}{2(1-T)}+\frac{1}{2}\cdot\frac{2}{1-T}\cdot\frac{1}{1+T-(1-T)z^{-1}}$$

$$=-\frac{1}{2(1-T)}+\frac{1}{1-T}\cdot\frac{1}{1+T-(1-T)z^{-1}}$$

$$=-\frac{1}{2(1-T)}+\frac{1}{(1-T)(1+T)}\cdot\frac{1}{1-\frac{1-T}{1+T}z^{-1}}$$

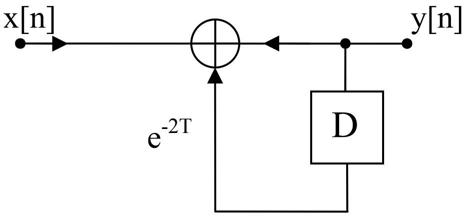
$$h_d[n]=-\frac{1}{2(1-T)}\delta[n]+\frac{1}{\left(1-T^2\right)}\cdot\left(\frac{1-T}{1+T}\right)^n\sigma[n]$$

Punctul d)



Pentru sistemul analogic

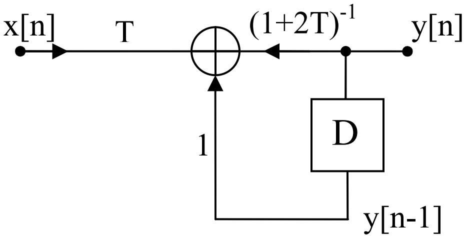
$$y+2\int y=\int x$$



Pentru sistemul numeric de la a)

$b_0 = 1;\ a_0 = 1;\ a_1 = -e^{-2T}$

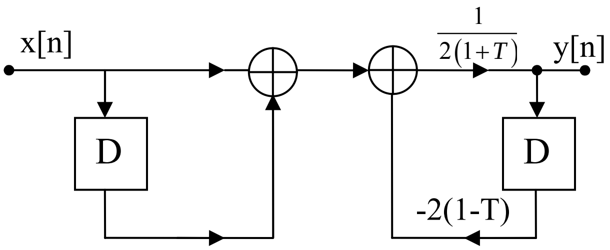
$y[n] - e^{-2T} y[n-1] = x[n]$



Pentru sistemul numeric de la b)

$b_0 = T;\ a_0 = 1 + 2T;\ a_1 = -1$

$(1+2T)\, y[n] - y[n-1] = Tx[n]$



Pentru sistemul numeric de la c)

$b_0 = 1;\ b_1 = 1;$

$a_0 = 2(1+T);\ a_1 = 2(1-T)$

$2(1+T)\, y[n] + 2(1-T)\, y[n-1] =$

$= x[n] + x[n-1]$

Punctul e)

$$x(t) = \sin t;\ \ y(t) = \left|H(1)\right| \sin\left(t + \arg\{H(1)\}\right)$$

$$H(s) = \frac{1}{s+2}\ \Rightarrow H(\omega) = \frac{1}{2 + j\omega}\ \Rightarrow H(1) = \frac{1}{2 + j}\ \Rightarrow H(1) = \frac{1}{\sqrt{5}}$$

$$\arg\{H(1)\} = -\arg\{2 + j\} = -\operatorname{arctg}\frac{1}{2}\ \Rightarrow y(t) = \frac{1}{\sqrt{5}} \sin\left(t - \operatorname{arctg}\frac{1}{2}\right)$$

$$\Rightarrow A_y = \frac{1}{\sqrt{5}}$$

Pentru sistemele numerice:

$$x[n] = \sin(0{,}1n) = \sin\left(\frac{n}{10}\right) \Rightarrow y[n] = \left|H_d\left(\frac{1}{10}\right)\right| \sin\left(\frac{n}{10} + \arg\left\{H_d\left(\frac{1}{10}\right)\right\}\right)$$

Pentru sistemul numeric de la a)

$$H_d(z) = \frac{1}{1 - e^{-2T} \cdot z^{-1}}\ \Rightarrow H_d(\Omega) = \frac{1}{1 - e^{-2T} \cdot z^{-j\Omega}} = \frac{1}{1 - e^{-2T} \cos\Omega + j \cdot e^{-2T} \sin\Omega}$$

$$\Rightarrow \left|H_d(\Omega)\right| = \frac{1}{\sqrt{\left(1-e^{-2T}\cos\Omega\right)^2 + e^{-4T}\sin^2\Omega}} = \frac{1}{\sqrt{1+e^{-4T}-2e^{-2T}\cos\Omega}}$$

$$\left|H_d\left(\frac{1}{10}\right)\right| = \frac{1}{\sqrt{1+e^{-0,4}-2e^{-0,2}\cos\frac{1}{10}}} \cong \frac{1}{\left(1-e^{-0,2}\right)^2}$$

Pentru sistemul numeric de la b)

$$H_d(z) = \frac{T}{1+2T\cdot z^{-1}} \Rightarrow H_d(\Omega) = \frac{T}{1+2T\cdot e^{-j\Omega}} = \frac{T}{1+2T\cos\Omega - j\sin\Omega}$$

$$\Rightarrow \left|H_d(\Omega)\right| = \frac{T}{\sqrt{\left(1+2T\cos\Omega\right)^2 + \left(2T\sin\Omega\right)^2}} = \frac{T}{\sqrt{1+4T^2+4T\cos\Omega}}$$

$$\Rightarrow \left|H_d\left(\frac{1}{10}\right)\right| \cong \frac{0,1}{\sqrt{1,44}} = \frac{0,1}{1,2} = 0,083$$

Pentru sistemul numeric de la c)

$$H_d(z) = \frac{1}{2}\cdot\frac{1+z^{-1}}{1+T-(1-T)z^{-1}} \Rightarrow H_d(\Omega) = \frac{1}{2}\cdot\frac{1+e^{-j\Omega}}{1,1-0,9e^{-j\Omega}}$$

$$\Rightarrow \left|H_d(\Omega)\right| = \frac{1}{2}\cdot\sqrt{\frac{\left(1+\cos\Omega\right)^2+\sin^2\Omega}{\left(1,1-0,9\cos\Omega\right)^2+\left(0,9\sin\Omega\right)^2}} = \frac{1}{2}\cdot\sqrt{\frac{2+2\cos\Omega}{1,21+0,81-1,98\cos\Omega}}$$

Se dă lui Ω valoarea 0,1 si se calculează.

Problema 5. Fie un sistem în timp continuu cu răspunsul la impuls $h_c(t)$ şi funcția de transfer $H_c(s)$. Ecuația diferențială care descrie acest sistem este:

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x}{dt^k}$$

Se aproximează acest sistem cu unul în timp discret folosind aproximarea:

$$\left.\frac{dx}{dt}\right|_{t=nT} \cong \frac{x[n+1]-x[n]}{T}$$

Se definesc diferențele de ordinul 0 : $\nabla^{(0)}\left\{x[n]\right\} = x[n]$

de ordinul 1 : $\nabla^{(1)}\left\{x[n]\right\} = \dfrac{x[n+1]-x[n]}{T}$

şi de ordinul k : $\nabla^{(k)}\left\{x[n]\right\} = \nabla^{(k-1)}\left\{\nabla^{1}\left\{x[n]\right\}\right\}$

a) Fie $H_c(s)=\dfrac{s}{(s+1)^2}$. Să se determine funcția de transfer a sistemului discret echivalent $H_d(z)$;

b) Care este legătura dintre $H_c(s)$ şi $H_d(z)$?

Rezolvare

$$\nabla^{(1)}\{x[n]\}=\frac{x[n+1]-x[n]}{T}$$

$$\nabla^{2}\{x[n]\}=\nabla^{(1)}\left\{\frac{x[n+1]-x[n]}{T}\right\}$$

$$=\frac{1}{T}\left[\nabla^{(1)}\{x[n+1]\}-\nabla^{(1)}\{x[n]\}\right]$$

$$=\frac{1}{T}\left[\frac{x[n+2]-x[n+1]}{T}-\frac{x[n+1]-x[n]}{T}\right]$$

$$=\frac{1}{T^2}\left[x[n+2]-2x[n+1]+x[n]\right]$$

Se observă că:

$$\nabla^{k}\{x[n]\}=\frac{1}{T^k}\sum_{l=0}^{k}C_k^l x[n+l](-1)^l$$

Punctul a)

$$H_c(s)=\frac{s}{s^2+2s+1}=\frac{b_1 s}{a_0+a_1 s+a_2 s^2}$$

$$\Rightarrow b_0=0;\ b_1=1;\ a_0=1;\ a_1=2;\ a_2=1$$

Ecuatia diferențială corespunzătoare este :

$$y(t)+2\frac{dy(t)}{dt}+\frac{d^2y(t)}{dt^2}=\frac{dx(t)}{dt}$$

Ecuația cu diferențe finite corespunzătoare este :

$$y[n]+2\nabla^{(1)}\{y[n]\}+\nabla^{(2)}\{y[n]\}=\nabla^{(1)}\{x[n]\}$$

Adică :

$$y[n]+2\frac{y[n+1]-y[n]}{T}+\frac{1}{T^2}\left[y[n+2]-2y[n+1]+y[n]\right]=\frac{x[n+1]-x[n]}{T}$$

Sau :

$$T^2y[n]+2T\left(y[n+1]-y[n]\right)+y[n+2]-2y[n+1]+y[n]=Tx[n+1]-Tx[n]$$

Luând în ambii membrii ai ultimei relații transformata Z, se obține:

$$T^2Y[z]+2T(z-1)Y[z]+z^2Y[z]-2zY[z]+Y[z]=TzX[z]-TX[z]$$

Sau :

$$Y(z)\left[T^2+2Tz-2T+z^2-2z+1\right]=X(z)(Tz-T)$$

$$\Rightarrow H_d(z)=\frac{Tz-T}{z^2+2(T-1)z+T^2-2T+1}$$

Punctul b)

$$H_d(z)=\frac{Tz-T}{(z+T-1)^2}=\frac{T(z-1)}{(z+T-1)^2}$$

$$=\frac{T(z-1)}{T^2\left(\frac{z-1}{T}+1\right)^2}=\frac{1}{T}\cdot T\frac{\frac{z-1}{T}}{\left(\frac{z-1}{T}+1\right)^2}$$

Deci : $H_d(z)=\dfrac{\frac{z-1}{T}}{\left(\frac{z-1}{T}+1\right)^2}$

Se constată că : $H_d(z)=H_c(s)\big|_{s=\frac{z-1}{T}}$